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LETTER TO THE EDITOR

The disappearance of spontaneous magnetisation in the Ising model with even-spin interactions at high temperatures

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Abstract. We prove that an Ising model with even-spin interactions has no spontaneous magnetisation at high temperatures when the ranges of interactions are finite.

It was proved by Griffiths (1967b), in terms of inequalities for correlation functions, that a ferromagnetic Ising model with two-spin interactions has no spontaneous magnetisation at high temperatures, namely at temperatures above the mean-field approximation to the Curie temperature. Fisher (1967) proved independently, in terms of the generating functions for self-avoiding random works, that the mean-field and Bethe approximations yield upper bounds to the Curie temperature. An improvement for an upper bound to the Curie temperature had been made, in terms of inequalities for correlation functions, by Thompson (1971) and Krinsky (1975) and others. On the other hand, these works were extended to other spin models with two-spin interactions by, e.g., Pearce (1981) and Tasaki and Hara (1984). Their work was restricted to systems with only two-spin interactions. In a recent paper (Morita and Horiguchi 1985) we proved, by using the nature of the Cayley tree, the non-existence of the spontaneous magnetisation in an Ising model with two- and four-spin interactions on the square lattice at high temperatures and we gave a discussion for a general system. In this letter, we use an inverse decimation (or a decoration) transformation and Griffiths' results for the system of two-spin interactions to prove that a ferromagnetic Ising model with only even-spin interactions has no spontaneous magnetisation at high temperatures. We have found a better upper bound to the Curie temperature for the Ising model with two- and four-spin interactions on the square lattice than that obtained previously (Morita and Horiguchi 1985).

We consider an Ising spin model on a finite set Λ of a d-dimensional lattice Z^d whose Hamiltonian contains only even-spin interactions except for an external field term. We assume that the maximum number of spins which interact with each other is $2n$. The Hamiltonian is given as follows

$$H_{2n} = H_1 - \sum_{\substack{P\subset\Lambda \\ |P|\leq 2n}} J_P\sigma_P, \tag{1}$$

where H_1 is the external field term: $H_1 = -h\sum_{i\in\Lambda}\sigma_i$ and h is a uniform external field. For a subset P of Λ, we define σ_P as $\sigma_P = \prod_{i\in P}\sigma_i$. J_P is an interaction constant for the spins on sites belonging to P and assumed to be non-negative. We assume that

J_P is zero unless $|P|$ is even and $|P| \leq 2n$, where $|A|$ denotes the cardinality of set A. Furthermore, we assume that it is possible to find a number r such that J_P is zero whenever diam $(P) > r$, namely, the ranges of interactions are finite. We define the spontaneous magnetisation:

$$\langle \sigma_i \rangle = \lim_{h \downarrow 0} \lim_{|\Lambda| \uparrow \infty} \langle \sigma_i \rangle_{H_{2n}}. \tag{2}$$

In (2) and in the following equations, the angular brackets with H denote the canonical average of the quantity between them in the system described by the Hamiltonian H.

We take up terms of $2n$-spin interactions in H_{2n}. Let us separate them from the other parts in H_{2n} and rewrite H_{2n} as follows

$$H_{2n} = H_1 - \sum_{\substack{P \subset \Lambda \\ |P| \leq 2n-2}} J_P \sigma_P - \sum_j J_{A(j)} \sigma_{j_1} \sigma_{j_2} \ldots \sigma_{j_{2n}}, \tag{3}$$

where a set $\{j_1, j_2, \ldots, j_{2n}\}$ is labelled by a number j and $A(j)$ is used to denote the set. The last summation in (3) is taken over the index set for $\{A(j)\}$. We introduce an auxiliary Hamiltonian $H_{2n-2}^{(a)}$ with ghost spins $\{\sigma_j\}$ on ghost sites $\{j\}$:

$$H_{2n-2}^{(a)} = H_1 - \sum_{\substack{P \subset \Lambda \\ |P| \leq 2n-2}} J_P \sigma_P - \sum_j K_{A(j)} \sigma_j (\sigma_{j_1} \sigma_{j_2} \ldots \sigma_{j_n} + \sigma_{j_{n+1}} \sigma_{j_{n+2}} \ldots \sigma_{j_{2n}}) \tag{4}$$

where $K_{A(j)}$ is an effective ferromagnetic interaction constant related to $J_{A(j)}$ as follows:

$$\beta K_{A(j)} = \tfrac{1}{2} |\cosh^{-1} \exp(2\beta J_{A(j)})|, \tag{5}$$

where β is $1/kT$ as usual. Then we have

$$\langle \sigma_i \rangle_{H_{2n}} = \langle \sigma_i \rangle_{H_{2n-2}^{(a)}}. \tag{6}$$

$K_{A(j)}$ is non-negative and at high temperatures $\beta K_{A(j)} = (\beta J_{A(j)})^{1/2} \times (1 + \mathrm{O}(\beta J_{A(j)}))$. We call this procedure an inverse decimation transformation. We note here that, in general, a k-spin interaction with $k \geq 2$ is expressed in terms of a k_1-spin interaction and a k_2-spin interaction, where $k_1 + k_2 = k + 2$ and both k_1 and k_2 are greater than or equal to 2.

When n is odd, we regard the ghost sites as a part of real sites and the ghost spins as a part of real spins. Then it amounts to considering the Ising model on the set $\Lambda \cup \{j\}$. The maximum number of spins which participate an interaction in $H_{2n-2}^{(a)}$ is at most $2n-2$ and all of the interactions in $H_{2n-2}^{(a)}$ are even-spin interactions except the external field term.

When n is even, we have odd-spin interactions $\{K_{A(j)} \sigma_j \sigma_{j_1} \sigma_{j_2} \ldots \sigma_{j_n}, K_{A(j)} \sigma_j \sigma_{j_{n+1}} \sigma_{j_{n+2}} \ldots \sigma_{j_{2n}}\}$ in $H_{2n-2}^{(a)}$. We apply a positive uniform external field $\tilde{h}$ to the ghost sites

$$H_{2n-2,\tilde{h}}^{(a)} = H_{2n-2}^{(a)} - \tilde{h} \sum_j \sigma_j. \tag{7}$$

Then, by using the Griffiths-Kelly-Sherman inequalities (Griffiths 1967a, Kelly and Sherman 1968), we have

$$\langle \sigma_i \rangle_{H_{2n-2}^{(a)}} \leq \langle \sigma_i \rangle_{H_{2n-2,\tilde{h}}^{(a)}}. \tag{8}$$

By letting $\tilde{h}$ tend to infinity, we have

$$\langle \sigma_i \rangle_{H_{2n-2}^{(a)}} \leq \langle \sigma_i \rangle_{\tilde{H}_{2n-2}}, \tag{9}$$

where

$$\tilde{H}_{2n-2}=H_1-\sum_{\substack{P\subset\Lambda\\|P|\leqslant 2n-2}} J_P\sigma_P-\sum_j K_{A(j)}(\sigma_{j_1}\sigma_{j_2}\ldots\sigma_{j_n}+\sigma_{j_{n+1}}\sigma_{j_{n+2}}\ldots\sigma_{j_{2n}}). \tag{10}$$

In this way, we have a system whose Hamiltonian contains only even-spin interactions except H_1. The maximum number of spins which participate in an interaction in $\tilde{H}_{2n-2}$ is at most $2n-2$.

Now, applying the same procedure to the system $H^{(a)}_{2n-2}$ or $\tilde{H}_{2n-2}$, we will obtain a system whose Hamiltonian contains only even-spin interactions except H_1 and the maximum number of spins which participate in an interaction is at most $2n-4$. Applying this procedure successively, we finally arrive within $n-1$ times at the system on a lattice $\tilde{\Lambda}$ consisting of Λ and ghost sites, in which the Hamiltonian can be written as follows

$$\tilde{H}^{(a)}_2=H_1-\sum_{k,l\in\tilde{\Lambda}}\tilde{J}_{k,l}\sigma_k\sigma_l. \tag{11}$$

Each of $\tilde{J}_{k,l}$, which is contained in H_{2n} or obtained from J_P in H_{2n}, is non-negative. Those obtained from J_P with $|P|\geqslant 4$ goes to $\beta^{-1}(\beta J_P)^{1/m}$, where $m=2^{\nu-1}$ when $2^{\nu-1}<|P|\leqslant 2^{\nu}$, as β goes to zero. The range of $\tilde{J}_{k,l}$ is finite because in each inverse decimation transformation the range of interaction does not increase. For $i\in\Lambda$, we have

$$\langle\sigma_i\rangle_{H_{2n}}\leqslant\langle\sigma_i\rangle_{\tilde{H}^{(a)}_2}. \tag{12}$$

As (12) holds for any subset Λ of Z^d and any positive value of h and both members of (12) exist in the thermodynamic limit, we have

$$\lim_{h\downarrow 0}\lim_{|\Lambda|\uparrow\infty}\langle\sigma_i\rangle_{H_{2n}}\leqslant\lim_{h\downarrow 0}\lim_{|\Lambda|\uparrow\infty}\langle\sigma_i\rangle_{\tilde{H}^{(a)}_2}. \tag{13}$$

However, the right-hand side is known to be zero at high temperatures (Griffiths 1967a). Thus we have from (2) that there is no spontaneous magnetisation in the system described by H_{2n} at high temperatures.

Here we make a remark on the inverse decimation transformation. There are many possibilities of dividing $2n$ spins into two groups each of which contains two or more spins. Two typical cases are as follows. When $2n$ spins are divided into two groups with even spins then we have an inequality similar to the one obtained from (6), (8) and (9) where the auxiliary Hamiltonian does not contain any ghost spin. This procedure can be carried out until the maximum number of spins contained in an auxiliary Hamiltonian is two. When $2n$ spins are divided into two groups with odd spins more than two, then we have an equality similar to (6). This procedure can be carried out until the maximum number of spins contained in an auxiliary Hamiltonian is four. Finally each of the groups of four spins is divided into two groups with two spins and we have an inequality similar to the one obtained from (6), (8) and (9). An upper bound to the Curie temperature depends on these divisions but the disappearance of the spontaneous magnetisation at high temperatures holds regardless of the divisions.

As an example, we consider an Ising model with ferromagnetic two-spin and four-spin interactions on the square lattice. The Hamiltonian is given as follows

$$H_4=-h\sum_{(i,j)}\sigma_{i,j}-J^x_2\sum_{(i,j)}\sigma_{i,j}\sigma_{i+1,j}-J^y_2\sum_{(i,j)}\sigma_{i,j}\sigma_{i,j+1}-J_4\sum_{(i,j)}\sigma_{i,j}\sigma_{i+1,j}\sigma_{i,j+1}\sigma_{i+1,j+1}, \tag{14}$$

where (i, j) denotes a lattice site of the square lattice. We define an auxiliary Hamiltonian

$$\tilde{H}_2 = -h \sum_{(i,j)} \sigma_{i,j} - J_2^x \sum_{(i,j)} \sigma_{i,j}\sigma_{i+1,j} - J_2^y \sum_{(i,j)} \sigma_{i,j}\sigma_{i,j+1} - 2K_4 \sum_{(i,j)} \sigma_{i,j}\sigma_{i+1,j}, \tag{15}$$

where

$$K_4 = J_4 + \tfrac{1}{2}kT \ln[1 + (1 - e^{-4\beta J_4})^{1/2}]. \tag{16}$$

We have

$$\langle\sigma_i\rangle_{H_4} \leq \langle\sigma_i\rangle_{\tilde{H}_2}. \tag{17}$$

Thus we have no spontaneous magnetisation in the system of H_4 at $T \geq T_c^{(u)}$ which is determined by the following equation

$$\sinh[2\beta(J_2^x + 2K_4)] \sinh(2\beta J_2^y) = 1. \tag{18}$$

For $J_2^x = J_2^y = J_2$, we show in figure 1 the obtained upper bound $T_c^{(u)}$ to the Curie temperature as a function of J_2/J_4.

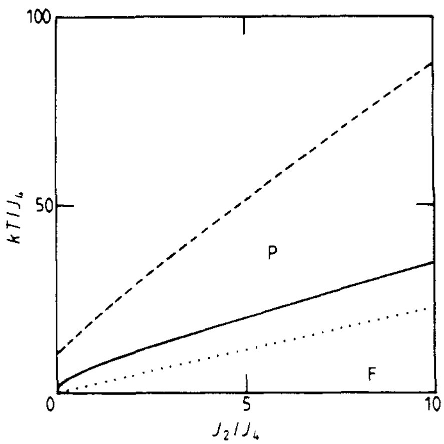

Figure 1. We show the upper bounds to the Curie temperature for an Ising model with two-spin and four-spin interactions on the square lattice. The full curve is the one obtained in the present paper. The broken curve is the one obtained in a previous paper (Morita and Horiguchi 1985). The dotted curve is a lower bound obtained by putting J_4 zero to the Curie temperature; the system has a spontaneous magnetisation below this line.

Finally, we make three remarks. The first one is for a system with multiple-spin interactions in general, namely with even-spin interactions and also odd-spin interactions. By using the inverse decimation transformation repeatedly, we obtain the following equality

$$\langle\sigma_i\rangle_H = \langle\sigma_i\rangle_{\tilde{H}^{(a)}}. \tag{19}$$

Here H is the Hamiltonian of the original system and $\tilde{H}^{(a)}$ is a Hamiltonian obtained from H. $\tilde{H}^{(a)}$ consists of the external field term, two-spin interactions and three-spin interactions. Disappearance of the spontaneous magnetisation in the system of $\tilde{H}^{(a)}$ depends on the three-spin interactions. The second one is for an Ising model of spin

$p/2$ which is greater than $1/2$. From the results obtained by Griffiths (1969), an Ising spin of spin $p/2$ is expressed in terms of a cluster of p Ising spins of $1/2$ interacting among themselves through suitable ferromagnetic two-spin interactions. Then the results obtained in this letter hold in the Ising model of spin $p/2$ with even-spin interactions. The final one is for an Ising model with ferromagnetic and also antiferromagnetic even-spin interactions. We can use theorems 1 and 2 which we proved in a previous paper (Horiguchi and Morita 1979), and then there is no spontaneous magnetisation in the system at high temperatures.

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